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Covariant formulation of field theories associated with p-branes

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Abstract

We discuss the covariant formulation of local field theories described by the companion Lagrangian associated with p-branes. The covariantization is shown to be useful for clarifying the geometrical meaning of the field equations and also their relation to the Hamilton–Jacobi formulation of the standard Dirac–Born–Infeld theory.

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1. Introduction

A significant class of equations of motion occurring in physics have the property of general covariance, i.e. the property that the solutions of these equations remain solutions under a large set of transformations. The best known examples are the equations of general relativity, Yang–Mills and the Maxwell equations in terms of the gauge fields. The fact, realized by the cognoscenti, that the origin of covariance in those examples could be unified by the construction of a covariant derivative became generally known in the mid 1970s with the adoption of fibre bundle language in the discussion of connections. In this paper we discuss a further example of the equations of motion arising from what we have called the companion Lagrangian, which may be considered as a continuation of the Dirac–Born–Infeld (DBI) equations describing D-branes to the situation where the target space is of smaller dimension than the base space. The genesis of these equations lies in the idea of replicating for strings and branes the situation in ordinary quantum mechanics in which the classical point particle Lagrangian is replaced by the quantum Klein–Gordon Lagrangian [1–5].

Let ϕ^i be n fields each dependent upon coordinates x^μ ($\mu = 1, 2, \ldots, d > n$) of the base space. Then, in its simplest form, the companion Lagrangian $\mathcal{L}$ is

$$\mathcal{L} = \sqrt{\det \left| \frac{\partial \phi^i}{\partial x_\mu} \frac{\partial \phi^j}{\partial x_\mu} \right|}. \tag{1}$$

The simplest example of an equation arising from this for $n = 1,\ d = 2$ is the Bateman equation,

$$\left(\frac{\partial\phi}{\partial x_1}\right)^2 \frac{\partial^2\phi}{\partial x_2^2} + \left(\frac{\partial\phi}{\partial x_2}\right)^2 \frac{\partial^2\phi}{\partial x_1^2} - 2\left(\frac{\partial\phi}{\partial x_1}\right)\left(\frac{\partial\phi}{\partial x_2}\right)\frac{\partial^2\phi}{\partial x_1 \partial x_2} = 0 \qquad (2)$$

references [6, 7] and for $n = 1$, arbitrary d the companion equation is a sum of $\binom{d}{2}$ such Bateman expressions set to zero. It is known that the solution of (2) is given implicitly by solving the equation,

$$x_1 F(\phi(x)) + x_2 G(\phi(x)) = c \qquad (3)$$

where F, G are arbitrary functions and c is a constant. From the form of the solution, it is obvious that the Bateman equation is invariant under any change of $\phi \to \phi'(\phi)$. Another example is for two fields $\phi,\ \psi$ in three dimensions, in which the companion equation can be recast in the form,

$$\det \begin{vmatrix} 0 & 0 & \phi_1 & \phi_2 & \phi_3 \\ 0 & 0 & \psi_1 & \psi_2 & \psi_3 \\ \phi_1 & \psi_1 & \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_2 & \psi_2 & \phi_{12} & \phi_{22} & \phi_{23} \\ \phi_3 & \psi_3 & \phi_{13} & \phi_{23} & \phi_{33} \end{vmatrix} = 0 \qquad (4)$$

with a similar equation with second derivatives of ψ, where subscripts denote derivatives, e.g., $\phi_\mu = \partial\phi/\partial x^\mu$. These equations have been studied in [7] as one of the universal field equations and shown to be covariant under any arbitrary redefinition of the fields. This remarkable property also holds for the general (n, d) case, i.e. *the companion equation is invariant under any change of fields* $\phi^i \to \phi'^i(\phi^j)$, which will be transparent in the following covariant formulation. This symmetry corresponds to the general reparametrization invariance in the DBI formulation of $p = n - 1$ branes, where the theory with d fields $X^\mu(\tau^i)$ is invariant under reparametrization of the n world volume coordinates τ^i.

In the next section, we reconsider the companion theory in a manifestly covariant way and clarify the geometrical meaning of the companion equations. In section 3, we study the relation between the Hamilton–Jacobi (HJ) formulation of the DBI theory and the companion theory, showing that the latter possesses a class of solutions of the former, which are characterized by a divergence-free condition of a degenerate metric defined in the latter theory. The relation between the DBI and companion theories is demonstrated in section 4 explicitly in the particle $(n = 1)$ case in an arbitrary number of dimensions.

2. Covariant formulation

2.1. Notation

For simplicity, we work in d-dimensional Euclidean space with the flat metric $\delta_{\mu\nu}$ and the totally antisymmetric tensor $\epsilon_{\nu_1\nu_2\ldots\nu_d}$ with $\epsilon_{12\ldots d} = +1$. Indices with an arrow above them represent the set of several indices. $\vec{\mu}, \vec{\nu}, \vec{\rho}, \vec{\sigma}$ each have n components, e.g., $\vec{\mu} = \{\mu_1, \mu_2, \ldots, \mu_n\}$. $\vec{\tau}, \vec{\kappa}, \vec{\eta}$ each have $(d - n)$ components, e.g., $\vec{\kappa} = \{\kappa_1, \kappa_2, \ldots, \kappa_{d-n}\}$. If the (double) prime $'$ $('')$ is used for the indices, $\vec{\mu}, \vec{\nu}$ or $\vec{\tau}, \vec{\kappa}$, then their components start from the second (third) entry of un-primed ones, e.g., $\vec{\mu}' = \{\mu_2, \ldots, \mu_n\}$, $\vec{\kappa}' = \{\kappa_2, \ldots, \kappa_{d-n}\}$, $\vec{\mu}'' = \{\mu_3, \ldots, \mu_n\}$ and $\vec{\kappa}'' = \{\kappa_3, \ldots, \kappa_{d-n}\}$.

2.2. Jacobians

The companion equations may be expressed more succinctly in terms of the Jacobians, which are defined as

$$J_{\vec{\kappa}} = J_{\kappa_1\kappa_2\ldots\kappa_{d-n}} = \epsilon_{\kappa_1\kappa_2\ldots\kappa_{d-n}\nu_1\nu_2\ldots\nu_n}\phi^1_{\nu_1}\phi^2_{\nu_2}\ldots\phi^n_{\nu_n}$$
$$= \frac{1}{n!}\,\epsilon_{\vec{\kappa}\vec{\nu}}\,\epsilon_{i_1\ldots i_n}\phi^{i_1}_{\nu_1}\ldots\phi^{i_n}_{\nu_n} \equiv \frac{1}{n!}\,\epsilon_{\vec{\kappa}\vec{\nu}}\,\tilde{J}_{\vec{\nu}}. \tag{5}$$

The derivatives of the Jacobians are

$$\frac{\partial J_{\vec{\kappa}}}{\partial\phi^i_\mu} = \frac{1}{(n-1)!}\,\epsilon_{\vec{\kappa}\mu\vec{\nu}'}\,\epsilon_{ii_2\ldots i_n}\phi^{i_2}_{\nu_2}\ldots\phi^{i_n}_{\nu_n} \equiv \frac{1}{(n-1)!}\,\epsilon_{\vec{\kappa}\mu\vec{\nu}'}\,\tilde{J}_{i,\vec{\nu}'}. \tag{6}$$

Using the Jacobians, the companion Lagrangian (1) is written as

$$\mathcal{L} = \sqrt{\det\left|\frac{\partial\phi^i}{\partial x_\mu}\frac{\partial\phi^j}{\partial x_\mu}\right|} = \sqrt{\frac{1}{(d-n)!}J_{\vec{\kappa}}J_{\vec{\kappa}}} = \sqrt{\frac{1}{n!}\tilde{J}_{\vec{\mu}}\tilde{J}_{\vec{\mu}}} \tag{7}$$

from which the equation of motion is derived,

$$\frac{\partial^2\mathcal{L}}{\partial\phi^i_\mu\partial\phi^j_\nu}\phi^j_{\mu\nu} = \frac{1}{(d-n)!^2}\mathcal{L}^{-3}\left(J_{\vec{\tau}}J_{\vec{\tau}}\frac{\partial J_{\vec{\kappa}}}{\partial\phi^i_\mu}\frac{\partial J_{\vec{\kappa}}}{\partial\phi^j_\nu} - J_{\vec{\kappa}}\frac{\partial J_{\vec{\kappa}}}{\partial\phi^i_\mu}J_{\vec{\tau}}\frac{\partial J_{\vec{\tau}}}{\partial\phi^j_\nu}\right)\phi^j_{\mu\nu} = 0. \tag{8}$$

As shown in [8], using the identity of epsilon tensors,

$$\epsilon_{\mu\nu_2\nu_3\ldots\nu_d}\epsilon_{\rho_1\rho_2\ldots\rho_d} = \epsilon_{\rho_1\nu_2\nu_3\ldots\nu_d}\epsilon_{\mu\rho_2\ldots\rho_d} + \epsilon_{\rho_2\nu_2\nu_3\ldots\nu_d}\epsilon_{\rho_1\mu\rho_3\ldots\rho_d} + \cdots + \epsilon_{\rho_d\nu_2\nu_3\ldots\nu_d}\epsilon_{\rho_1\rho_2\ldots\rho_{d-1}\mu} \tag{9}$$

where the index μ is swapped with each index in the second epsilon, we can rewrite the equation as

$$\frac{\partial^2\mathcal{L}}{\partial\phi^i_\mu\partial\phi^j_\nu}\phi^j_{\mu\nu} = \frac{1}{(r+1)!(r-1)!}\mathcal{L}^{-3}\left(\frac{\partial J_{\vec{\kappa}}}{\partial\phi^i_\tau}\frac{\partial J_{\vec{\kappa}}}{\partial\phi^j_\tau}\right)J_{\mu\vec{\tau}'}J_{\nu\vec{\tau}'}\phi^j_{\mu\nu} = 0 \tag{10}$$

where $r = d - n$. Under the assumption $\det|\partial J_{\vec{\kappa}}/\partial\phi^i_\tau\,\partial J_{\vec{\kappa}}/\phi^j_\tau| \neq 0$, we obtain the companion equation,

$$J_{\mu\vec{\kappa}'}J_{\nu\vec{\kappa}'}\phi^i_{\mu\nu} = 0. \tag{11}$$

This equation may be interpreted as a sum of universal field equations [7]. It will be shown below that the left-hand side (LHS) of this equation appears as a covariant derivative acting on the field ϕ^i_μ.

For later use, we note two useful identities for the Jacobians,

$$\frac{\partial J_{\vec{\kappa}}}{\partial\phi^j_\mu}\phi^i_\mu = \delta^i_j\,J_{\vec{\kappa}} \tag{12}$$

$$\frac{\partial J_{\vec{\kappa}}}{\partial\phi^j_\nu}\phi^j_\mu = (r+1)\delta_{\mu[\nu}J_{\vec{\kappa}]} = (\delta_{\mu\nu}J_{\vec{\kappa}} - \delta_{\mu\kappa_1}J_{\nu\kappa_2\ldots\kappa_r} - \cdots - \delta_{\mu\kappa_r}J_{\kappa_1\ldots\kappa_{r-1}\nu}). \tag{13}$$

From the identities, we obtain

$$\frac{\partial\mathcal{L}}{\partial\phi^j_\mu}\phi^i_\mu = \delta^i{}_j\mathcal{L} \qquad \frac{\partial\mathcal{L}}{\partial\phi^j_\nu}\phi^j_\mu = \mathcal{L}\left(\delta_{\mu\nu} - \frac{1}{(r-1)!}\mathcal{L}^{-2}J_{\mu\vec{\kappa}'}J_{\nu\vec{\kappa}'}\right). \tag{14}$$

2.3. Induced metric

The field $\phi^i(x)$ is the mapping from the d-dimensional base space with the flat metric $\delta_{\mu\nu}$ to the n-dimensional space labelled by $\{\phi^i\}$. The induced metric G^{ij} on the ϕ-space is then defined as the pullback of $\delta^{\mu\nu}$,

$$G^{ij} = \phi^i_\mu \phi^j_\mu. \tag{15}$$

The metric $G^{ij} \to (\partial\phi'^i/\partial\phi^k)(\partial\phi'^j/\partial\phi^l)G^{kl}$ transforms under the reparametrization of ϕ^i, as anticipated. The inverse of G^{ij} can be obtained explicitly by using the identities (12) and (13),

$$G_{ij} = \frac{1}{(r+1)!}\mathcal{L}^{-2}\frac{\partial J_{\vec\kappa}}{\partial\phi^i_\mu}\frac{\partial J_{\vec\kappa}}{\partial\phi^j_\mu} = \frac{1}{(n-1)!}\mathcal{L}^{-2}\tilde J_{i,\vec\mu'}\tilde J_{j,\vec\mu'}. \tag{16}$$

Note that the square of the Lagrangian can be written as $\mathcal{L}^2 = \det|G^{ij}|$. Having obtained G_{ij}, we go back to the base space again with the induced metric $g_{\mu\nu}$, the pullback of G_{ij},

$$\begin{aligned} g_{\mu\nu} = \phi^i_\mu\phi^j_\nu G_{ij} &= \frac{1}{(n-1)!}\mathcal{L}^{-2}\tilde J_{\mu\vec\mu'}\tilde J_{\nu\vec\mu'} \\ &= \delta_{\mu\nu} - \frac{1}{(r-1)!}\mathcal{L}^{-2}J_{\mu\vec\kappa'}J_{\nu\vec\kappa'} \end{aligned} \tag{17}$$

where the identity (13) has been used in the last line. The metric $g_{\mu\nu}$ is manifestly invariant under the reparametrization of ϕ^i, although it is degenerate in our $d > n$ case. Note that $g_{\mu\nu}$ is the flat metric $\delta_{\mu\nu}$ for $d = n$, while it cannot be defined for $d < n$. Let us write down the metrics explicitly in the particle ($n = 1$) and string ($n = 2$) cases,

$$n = 1 \qquad g_{\mu\nu} = \frac{\phi_\mu\phi_\nu}{\phi_\lambda\phi_\lambda} \tag{18}$$

$$n = 2 \qquad g_{\mu\nu} = \frac{\tilde J_{\mu\rho}\tilde J_{\nu\rho}}{\det|\phi^i_\lambda\phi^j_\lambda|} = -\mathcal{L}^{-2}\begin{vmatrix} 0 & \phi^1_\nu & \phi^2_\nu \\ \phi^1_\mu & \phi^1_\lambda\phi^1_\lambda & \phi^1_\lambda\phi^2_\lambda \\ \phi^2_\mu & \phi^2_\lambda\phi^1_\lambda & \phi^2_\lambda\phi^2_\lambda \end{vmatrix}. \tag{19}$$

In these expressions, it is easy to see that ϕ^i_μ are eigenvectors of $g_{\mu\nu}$ with eigenvalue +1. This important property of $g_{\mu\nu}$ holds for general (n, d),

$$g_{\mu\nu}\phi^i_\nu = \phi^j_\mu G_{jk}G^{ki} = \phi^i_\mu \tag{20}$$

which leads us to define the projection operator $P_{\mu\nu} = (\delta_{\mu\nu} - g_{\mu\nu})$ acting on the d-dimensional vector space $\mathcal{V}$ in the base space. Then $\mathcal{V}$ is decomposed as the sum of two subspaces $\mathcal{V}_n$ and $\mathcal{V}_r$. The latter is the $(d - n)$-dimensional subspace orthogonal to the n vectors ϕ^i_μ, while the former is spanned by a linear combination of ϕ^i_μ. For an arbitrary $V_\mu \in \mathcal{V}$, we have

$$V_\mu = g_{\mu\nu}V_\nu + P_{\mu\nu}V_\nu = V_i\,\phi^i_\mu + P_{\mu\nu}V_\nu. \tag{21}$$

We introduce the dual vectors $Y_{j\nu} = \mathcal{L}^{-1}(\partial\mathcal{L}/\partial\phi^j_\nu) = G_{jk}\phi^k_\nu$ of ϕ^i_μ, which satisfy

$$\phi^i_\mu\,Y_{j\mu} = \delta^i_j \qquad \phi^i_\mu\,Y_{i\nu} = Y_{i\mu}\phi^i_\nu = g_{\mu\nu} \tag{22}$$

then the component V_i of the vector V_μ in (21) is expressed as $V_i = V_\mu Y_{i\mu}$. We will see in the next section that the connection for the reparametrization $\phi^i \to \phi'^i$ is constructed in terms of the fields ϕ^i_μ and $Y_{j\nu}$.

2.4. Induced connection

To construct the covariant formulation under the transformation $\phi^i \to \phi'^i(\phi^j)$, we first consider the geometry of the n-dimensional space labelled by coordinates $\{\phi^i\}$, with the metric $G^{(0)}_{ij}(\phi)$. The standard way to build covariant equations is to use the covariant derivative ∇_j, with a connection Γ^i_{jk},

$$\nabla_j V^i = \frac{\partial V^i}{\partial \phi^j} + \Gamma^i_{jk} V^k. \tag{23}$$

The assumption of the covariant constancy of the metric, $\nabla_j G^{(0)}_{ik} = 0$, and the torsion-free condition give the form of a Christoffel symbol to the connection Γ^i_{jk}. The pullback of Γ^i_{jk} onto the base space is obtained by acting with the factors $\phi^j_\nu \phi^k_\mu \phi^l_\lambda$ on it and using the relation $\phi^j_\nu \partial_j = \partial_\nu$,

$$\left(\phi^m_{\nu\mu} + \phi^j_\nu \phi^k_\mu \Gamma^m_{jk}\right) \phi^l_\lambda G^{(0)}_{lm} = \tfrac{1}{2}\left(\partial_\nu g^{(0)}_{\lambda\mu} + \partial_\mu g^{(0)}_{\lambda\nu} - \partial_\lambda g^{(0)}_{\nu\mu}\right) \equiv \tilde{\Gamma}_{\lambda\nu\mu}(g^{(0)}) \tag{24}$$

where $g^{(0)}_{\lambda\mu} = \phi^l_\lambda \phi^k_\mu G^{(0)}_{lk}$. The substitution of the induced metric $G^{ij} = \phi^i_\mu \phi^j_\mu$ into (24) leads us to the connection $K_\nu{}^i{}_k$,

$$K_\nu{}^i{}_k = \phi^j_\nu \Gamma^i_{jk}|_{G^{(0)}=G} = -Y_{k\mu} \tilde{\nabla}_\nu \phi^i_\mu \tag{25}$$

where $\tilde{\nabla}_\nu \phi^i_\mu = \phi^i_{\nu\mu} - \tilde{\Gamma}_{\lambda\nu\mu}(g)\, \phi^i_\lambda$, with $g_{\mu\nu}$, the degenerate metric in (17).

Then the induced derivative $\nabla_\nu V^i = \partial_\nu V^i + K_\nu{}^i{}_k V^k$ is manifestly covariant under the reparametrization of ϕ^i. We, however, note that the substitution $G^{(0)}_{ij} = G_{ij}$ in (25) cannot be justified since the induced metric G^{ij} is not a function of ϕ^i but ϕ^i_μ, where the relation $\phi^j_\nu \partial_j = \partial_\nu$ used above is incorrect when it acts on G^{ij}. Strictly speaking, the connection $K_\nu{}^i{}_k$ is *not* derived from Γ^i_{jk} in the ϕ^i-space but is *defined* in the analogy with $\phi^j_\nu \Gamma^i_{jk}(G^{(0)})$. Another remark is that $\tilde{\Gamma}_{\lambda\nu\mu}(g)$ in (25) looks like a connection in the base space. In fact, it can be shown that the derivative $\tilde{\nabla}_\nu \phi^i_\mu$ behaves as a tensor under the reparametrization $x^\mu \to x'^\mu$ preserving the form of the flat metric $\delta_{\mu\nu}$; $\partial_\lambda x'^\mu \partial_\rho x'^\nu \delta_{\lambda\rho} = \delta_{\mu\nu}$.

The covariant derivative ∇_ν acting on ϕ^i_μ becomes

$$\nabla_\nu \phi^i_\mu - \tilde{\Gamma}_{\lambda\nu\mu} \phi^i_\lambda = \partial_\nu \phi^i_\mu + K_\nu{}^i{}_k \phi^k_\mu - \tilde{\Gamma}_{\lambda\nu\mu} \phi^i_\lambda = P_{\mu\lambda} \tilde{\nabla}_\nu \phi^i_\lambda \tag{26}$$

or equivalently, using $P_{\mu\lambda} \phi^i_{\lambda\nu} = \partial_\nu g_{\mu\lambda} \phi^i_\lambda$, we have

$$\nabla_\nu \phi^i_\mu = \Gamma_{\lambda\nu\mu} \phi^i_\lambda \qquad \Gamma_{\lambda\nu\mu} = \partial_\nu g_{\mu\lambda} + g_{\mu\rho} \tilde{\Gamma}_{\lambda\nu\rho}. \tag{27}$$

The right-hand side (RHS) of (26) vanishes when it acts upon ϕ^j_μ, which gives

$$\nabla_\nu G^{ij} = 2\phi^{(i}_\mu \nabla_\nu \phi^{j)}_\mu = 2\phi^{(i}_\mu \tilde{\Gamma}_{\lambda\nu\mu} \phi^{j)}_\lambda = \phi^i_\lambda \phi^j_\mu \partial_\nu g_{\lambda\mu} \tag{28}$$

in which the RHS is actually zero due to the identity,

$$\partial_\nu g_{\lambda\mu} = (\delta_{\lambda\sigma} - g_{\lambda\sigma})\, Y_{k\mu} \phi^k_{\sigma\nu} + (\delta_{\mu\sigma} - g_{\mu\sigma})\, Y_{k\lambda} \phi^k_{\sigma\nu}. \tag{29}$$

Hence we obtain the metricity condition $\nabla_\nu G^{ij} = 0$ (and $\nabla_\nu G_{ij} = 0$) in the base space. The covariant derivative of $g_{\mu\nu}$ is also obtained from (27) and the identities $g_{\mu\rho} g_{\rho\nu} = g_{\mu\nu}$ and $\partial_\nu g_{\rho\lambda} - \tilde{\Gamma}_{\rho\nu\lambda} - \tilde{\Gamma}_{\lambda\nu\rho} = 0$,

$$\nabla_\lambda g_{\mu\nu} = \Gamma_{\rho\lambda\mu} g_{\rho\nu} + \Gamma_{\rho\lambda\nu} g_{\rho\mu} = \partial_\lambda g_{\mu\nu} \tag{30}$$

as anticipated since $g_{\mu\nu}$ is a scalar under the field redefinition of ϕ^i.

Finally, let us take the contraction $\mu = \nu$ in (27), then we have

$$\nabla_\mu \phi^i_\mu = \Gamma_{\lambda\mu\mu}\, \phi^i_\lambda = \partial_\mu g_{\mu\lambda}\, \phi^i_\lambda + g_{\mu\rho} \tilde{\Gamma}_{\lambda\mu\rho}\, \phi^i_\lambda. \tag{31}$$

The second term of the RHS vanishes due to the identity (29), which gives, with the formula of $g_{\mu\nu}$ in (17),

$$\nabla_\mu \phi^i_\mu = \partial_\mu g_{\mu\lambda}\,\phi^i_\lambda = P_{\mu\lambda}\phi^i_{\mu\lambda} = \frac{1}{(r-1)!}\mathcal{L}^{-2} J_{\mu\vec{\kappa}'} J_{\lambda\vec{\kappa}'}\phi^i_{\mu\lambda} = 0 \text{ (on shell)}. \tag{32}$$

Here, the companion equation (11) appears as the covariant derivative ∇_μ acting on ϕ^i_μ, which explicitly shows the general covariance of the companion equation.

3. DBI theory versus companion theory

The standard formulation of branes is given by a mapping $X^\mu(\tau^i)$ from the n-dimensional world volume to the d-dimensional target space. Let us consider the Lagrangian defined with derivatives of $X^\mu(\tau^i)$,

$$\mathcal{L}_p = (\det|g_{ij}|)^p \qquad g_{ij} = \frac{\partial X^\mu}{\partial \tau^i}\frac{\partial X^\mu}{\partial \tau^j}. \tag{33}$$

The equation of motion for $\mathcal{L}_p$ is written, as in [9], in terms of g_{ij} and the Christoffel symbol $\Gamma^k_{ij} = g^{km}\partial_m X^\lambda\,\partial_i\partial_j X^\lambda$,

$$g^{ij}\partial_i\partial_j X^\mu - \partial_i X^\mu g^{jk}\Gamma^i_{jk} + (2p-1)\partial_i X^\mu g^{ij}\Gamma^k_{jk} = 0. \tag{34}$$

Contraction of this equation with $\partial_l X^\mu$ yields

$$(2p-1)\Gamma^k_{lk} = \left(1 - \frac{1}{2p}\right)\mathcal{L}_p^{-1}\partial_l \mathcal{L}_p = 0 \tag{35}$$

and the other terms cancel. As in the previous discussion either $p = \frac{1}{2}$, or else $\partial_l \mathcal{L}_p = 0$. This leaves as equations of motion in all cases

$$g^{ij}\partial_i\partial_j X^\mu - \partial_i X^\mu g^{jk}\Gamma^i_{jk} = (\delta_{\mu\lambda} - g^{\mathrm{DBI}}_{\mu\lambda})\,g^{ij}\partial_i\partial_j X^\lambda = 0 \tag{36}$$

where the degenerate metric $g^{\mathrm{DBI}}_{\mu\lambda} = g^{kl}\partial_k X_\mu \partial_l X_\lambda$. The second equation explicitly shows, via the identity $g^{\mathrm{DBI}}_{\lambda\mu}\partial_l X^\mu = \partial_l X^\lambda$, that the number of independent equations of motion is $d - n$. In the following, we will concentrate on the DBI Lagrangian $\mathcal{L}_{\mathrm{DBI}} = M^n \mathcal{L}_{1/2}$, with the mass parameter M.

As is known in the particle case ($n = 1$), we introduce a field $\phi(x)$ as a HJ function for $\mathcal{L}_{\mathrm{DBI}}$, which gives the canonical conjugate momentum of $X^\mu(\tau)$ via the formula $p_\mu(\tau) = \partial_\mu\phi(x = X(\tau))$. The HJ equation is then obtained from the constraint for p_μ as $p^2 - M^2 = (\partial_\mu\phi)^2 - M^2 = 0$. The generalization of the HJ formulation to string and brane cases has been discussed in [4, 5, 10–13]. In their paper [5], Hosotani and Nakayama gave a simple derivation of the HJ equation for general $n > 1$ cases. They start with the DBI (Nambu–Goto) action for the string ($n = 2$);

$$I_2 = M^2 \int \mathrm{d}\tau\,\mathrm{d}\sigma \sqrt{\det|g_{ij}|} = M^2 \int \mathrm{d}\tau\,\mathrm{d}\sigma \sqrt{\frac{1}{2}\left(\frac{\partial(X^\mu, X^\nu)}{\partial(\sigma,\tau)}\right)^2}. \tag{37}$$

The covariant momentum tensor $p_{\mu\nu}$ given by

$$p_{\mu\nu} = \frac{M^2}{2}\frac{\frac{\partial(X^\mu,X^\nu)}{\partial(\sigma,\tau)}}{\sqrt{\frac{1}{2}\left(\frac{\partial(X^\mu,X^\nu)}{\partial(\sigma,\tau)}\right)^2}} \qquad p_{\mu\nu}p^{\mu\nu} = \frac{M^4}{2} \tag{38}$$

satisfies the equation of motion

$$\frac{\partial(p_{\mu\nu}, X^\nu)}{\partial(\sigma,\tau)} = 0 \tag{39}$$

which is an alternative form of equation (36). One may think of the solutions for the X^μ as being functions of $\sigma,\ \tau$ with $d-2$ additional parameters $\varphi_3, \ldots, \varphi_d$. Then $p_{\mu\nu}(\sigma, \tau; \varphi_a)$ can be considered as a function of the X^μ. Following Hosotani and Nakayama, we choose the parameters $\sigma,\ \tau$ in such a way that the area element of the world sheet with fixed φ_a is

$$4M^{-2} p^{\mu\nu}\, \mathrm{d}\sigma\, \mathrm{d}\tau = \mathrm{d}X^\mu\, \mathrm{d}X^\nu. \tag{40}$$

Choosing $\phi^1 = M\sigma$ and $\phi^2 = M\tau$, we obtain the relation between $\partial_i X^\mu$ and $\partial_\mu \phi^i$, $p_{\mu\nu} = \tilde{J}_{\mu\nu} = \partial_\mu\phi^1\partial_\nu\phi^2 - \partial_\nu\phi^1\partial_\mu\phi^2$, which gives the HJ equation for strings,

$$\frac{1}{2}\tilde{J}_{\mu\nu}\tilde{J}_{\mu\nu} = (\partial_\mu\phi^1)^2(\partial_\nu\phi^2)^2 - (\partial_\mu\phi^1\partial_\mu\phi^2)^2 = \frac{M^4}{4}. \tag{41}$$

It is easily seen that the equation of motion (39), thanks to the Bianchi identity for Jacobians $\partial_{[\lambda}\tilde{J}_{\mu\nu]} = 0$, is derived from the HJ equation as

$$\frac{\partial(p_{\mu\nu}, X^\nu)}{\partial(\sigma, \tau)} = \partial_\lambda \tilde{J}_{\mu\nu}\frac{\partial(X^\lambda, X^\nu)}{\partial(\sigma, \tau)} = 4M^{-2}\partial_\lambda\tilde{J}_{\mu\nu}\tilde{J}_{\lambda\nu} = M^{-2}\partial_\mu(\tilde{J}_{\lambda\nu}\tilde{J}_{\lambda\nu}) = 0. \tag{42}$$

These results can be generalized straightforwardly to membrane and general $p = n-1$-brane cases,

$$I_n = M^n \int d^n\tau \sqrt{\det|g_{ij}|} = M^n \int d^n\tau \sqrt{\frac{1}{n!}\left(\frac{\partial(X^{\mu_1}, \ldots, X^{\mu_n})}{\partial(\tau_1, \ldots, \tau_n)}\right)^2}. \tag{43}$$

The covariant momentum tensor $p_{\vec{\mu}} = p_{\mu_1\cdots\mu_n}$ is set to be equal to the Jacobian for n fields in (5),

$$p_{\vec{\mu}} = \frac{M^{2n}}{n!}\mathcal{L}_{\mathrm{DBI}}^{-1}\frac{\partial(X^{\mu_1}, \ldots, X^{\mu_n})}{\partial(\tau_1, \ldots, \tau_n)} = \epsilon_{i_1\ldots i_n}\phi^{i_1}_{\mu_1}\ldots\phi^{i_n}_{\mu_n} = \tilde{J}_{\vec{\mu}} \tag{44}$$

which leads to the HJ equation,

$$\frac{1}{n!}\tilde{J}_{\vec{\mu}}\tilde{J}_{\vec{\mu}} = \left(\frac{M^n}{n!}\right)^2. \tag{45}$$

The equation of motion $\partial(p_{\mu_1\mu_2\cdots\mu_n}, X^{\mu_2}, \ldots, X^{\mu_n})/\partial(\tau_1, \tau_2, \ldots, \tau_n) = 0$ is solved by the HJ equation and the Bianchi identity $\partial_{[\lambda}\tilde{J}_{\vec{\mu}]} = 0$ as in the string case. It is interesting to note that, under relation (44), the degenerate DBI metric in (36) turns out to be the companion metric (17),

$$g^{\mathrm{DBI}}_{\mu\lambda} = g^{kl}\partial_k X_\mu \partial_l X_\lambda = \frac{1}{(n-1)!}\left(\frac{n!}{M^n}\right)^2 p_{\mu\vec{\nu}'}p_{\lambda\vec{\nu}'} = g_{\mu\lambda}(X(\tau)). \tag{46}$$

As for the relation between the DBI and the companion theories, it is obvious that the HJ equation (45) in the former is the constancy condition of the Lagrangian (7), $\mathcal{L} = M^n/n!$, in the latter. Thus, any field configuration making the companion Lagrangian constant solves the DBI equation of motion. Here, let us consider solutions of the companion equation (11) in the configuration space of non-zero constant Lagrangian, which represent special points of the space to maintain the value of the Lagrangian (up to a total derivative) under any infinitesimal variation of fields ϕ^i. As shown in (32), the companion equation is proportional to $\nabla_\lambda\phi^i_\lambda = \partial_\lambda g_{\lambda\mu}\phi^i_\mu$, which would be regarded as a part of the decomposition (21) for $\partial_\lambda g_{\lambda\mu}$ into the n-dimensional subspace $\mathcal{V}_n$,

$$\nabla_\lambda g_{\lambda\mu} = \partial_\lambda g_{\lambda\mu} = V_i\,\phi^i_\mu + (\delta_{\mu\nu} - g_{\mu\nu})\partial_\lambda g_{\lambda\nu} \tag{47}$$

where $V_i = \partial_\lambda g_{\lambda\nu} Y_{i\nu} = G_{ij} \nabla_\nu \phi^j_\nu$. The second term in the RHS of (47) can be rewritten by using another decomposition of $\partial_\lambda g_{\lambda\nu} = \partial_\lambda (Y_{j\lambda}\phi^j_\nu) = \partial_\lambda Y_{j\lambda}\,\phi^j_\nu + \mathcal{L}^{-1}\partial_\nu \mathcal{L}$, where we used $Y_{j\lambda} = \mathcal{L}^{-1}(\partial\mathcal{L}/\partial\phi^j_\lambda)$, as

$$(\delta_{\mu\nu} - g_{\mu\nu})\,\partial_\lambda g_{\lambda\nu} = (\delta_{\mu\nu} - g_{\mu\nu})\,\mathcal{L}^{-1}\partial_\nu\mathcal{L} = 0 \qquad \text{if } \mathcal{L} = M^n/n! \tag{48}$$

showing that the subspace of HJ solutions given by the companion equation with constant Lagrangian is characterized geometrically by the divergence-free condition, $\nabla_\lambda g_{\lambda\mu} = \partial_\lambda g_{\lambda\mu} = 0$.

4. Solutions of companion and HJ equations

As has been remarked already, the companion equation (11) for a single field ($n = 1$) in d dimensions takes the form,

$$\sum_\mu \sum_{\nu\neq\mu} \left((\phi_\nu)^2\phi_{\mu\mu} - \phi_\mu\phi_\nu\phi_{\mu\nu}\right) = 0 \tag{49}$$

i.e. a sum of $\binom{d}{2}$ Bateman equations. A large class of solutions may be obtained by choosing d arbitrary functions $F^\mu(\phi)$ subject to the constraint,

$$\sum_\mu F_\mu(\phi)x_\mu = c = \text{const} \tag{50}$$

and solving this as an implicit equation for ϕ. This works because this equation implies that

$$\phi_\mu = \frac{-F_\mu}{F_\sigma{}' x_\sigma} \qquad \phi_{\mu\nu} = \frac{F_\mu{}' F_\nu + F_\nu{}' F_\mu}{\left(F_\sigma{}' x_\sigma\right)^2} - F_\mu F_\nu \frac{F_\lambda{}'' x_\lambda}{\left(F_\sigma{}' x_\sigma\right)^3} \tag{51}$$

where the prime denotes derivatives of F_μ with respect to ϕ. These results guarantee that (49) is satisfied. Solutions of this class may be extended to the case of more than one field in the following way. With an ansatz of the form,

$$\sum_\mu F^\mu(\phi^1)x_\mu = c^1 = \text{const} \qquad \sum_\mu G_\mu(\phi^2)x_\mu = c^2 = \text{const} \tag{52}$$

which are solved for ϕ^1 and ϕ^2, a similar reasoning shows that the companion equation of motion in an arbitrary number of dimensions, which is the sum of $\binom{d}{3}$ universal field equations for two fields in three dimensions, admits this implicit solution, which, in virtue of the covariance property, can be generalized by replacing the fields ϕ^1 and ϕ^2 by two arbitrary functions of them. This class of solutions may be generalized to an arbitrary number of fields in an obvious manner.

It can be easily seen that the set of companion solutions (50) for $n = 1$ contains configurations making the companion Lagrangian constant, i.e. solutions of the HJ equation. Assuming the constant c to be non-zero (and rescaled to one) and $F_\mu(\phi) = \beta_\mu F(\phi)$ with a constant vector β_μ, we have $F(\phi) = 1/\beta_\mu x_\mu$. On the other hand, the form of the function $F(\phi)$ is fixed by the condition $\mathcal{L} = M$, since it breaks the covariance of the companion equation under $\phi \to \phi'$,

$$\mathcal{L} = \sqrt{\phi_\mu\phi_\mu} = \sqrt{\frac{F_\mu F_\mu}{(F'_\nu x_\nu)^2}} = \sqrt{\beta^2}\frac{F^2}{F'} = M \tag{53}$$

where the HJ equation in the base space became a first-order differential equation in the ϕ-space, which is solved for $F(\phi)$ as $-M/\sqrt{\beta^2}\phi$. This leads to the standard solution of the HJ equation, $\phi = p_\mu x_\mu$, where $p_\mu = -M\beta_\mu/\sqrt{\beta^2}$ with $p^2 = M^2$.

In the above particle case, such a standard HJ solution can be easily obtained from the HJ equation itself. However, in the general p-brane cases, the companion equation equipped with the remarkably general covariance under field redefinitions would become a good starting point to find interesting HJ solutions. It is also intriguing to study the companion equations for general (n, d) in their own right as a possible extension of the solvable Bateman equation, which is now under progress.

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